A Finite Difference Method for One Dimensional Heat Equation

M. H. Kabir, A. Afroz, M. M. Hossain, M. O. Gani

Abstract—We consider one dimensional heat equation which is a Parabolic type of partial differential equations (PDE) as an initial boundary value problem (IBVP). The derivation of heat equation is presented here. We also present the derivation of a numerical scheme named explicit centered difference scheme for the heat equation. We develop a computer program to implement the scheme for the heat equation and present the numerical solution of the heat equation. We compare the analytic solution and numerical solution by error estimation.

Keywords— Heat Equation, initial boundary value problem (IBVP), finite difference method, partial differential equations (PDE).

1 INTRODUCTION

THE simplest model of heat flow is based on three principles: conservation of energy, Fouriers law of cooling and a constitutive law. The heat equation is characterized by the heat flow in terms of the measures of temperature. The heat equation is a second order partial differential equation that governs flow of heat within an object. We can see how the temperature changes with the time along an object by observing this equation. Let us note that the boundary value problems for the PDE belong to a large class of great importance problems in many scientific fields.

Since there are few papers devoted to the finite difference method named crank-nicolson scheme in Malgorzalaic [1]. Sharanjeet [2] found a numerical solution of one dimensional heat equation using cubic spline basis functions. In [2] one dimensional heat equation is solved using Galerkin B-spline finite element.

The aim of this article is to investigate an efficient finite difference scheme for one dimensional heat equation as an IBVP. In order to verify some qualitative behaviors of the scheme for one dimensional heat equation, we would like to make a comparative study between analytic solution and numerical solution of the heat equation.
In this paper we present the derivation of the heat equation based on Adam [9] and Raishinhania [5]. We study the analytic solution of this equation as an IBVP (Initial Boundary Value Problem) by the method of separation of variables from Gerald [3] and Raishinghania [5] . The derivation of the explicit finite difference scheme for heat equation as an IBVP is presented in section 4 based on Gerald [3], Smith [4] and Burden & Fares [7]. We develop a computer programming code to perform some numerical experiments and present relative error in section 5.

2 ONE DIMENSIONAL HEAT EQUATION

Here we present the derivation of the one dimensional heat equation based on Adam[9] and Raishinghania[5].

We consider the flow of heat by conduction in a thin rod made of a homogeneous material and perfectly insulated along its length so that heat can only flow through its ends. Any position along the rod is denoted by x, and the length of the rod is denoted by L (in meters) so that $0 \le x \le L$.

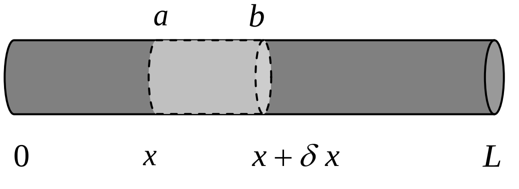

Fig 2.1: A thin rod of length L meters

Therefore, the temperature, $u(x,t)$ of the rod at any point is a function of position, x (in meters) and time t (in second). Suppose that the rod is raised to an assigned temperature distribution at time $t=0$ and then heat is allowed to flow by conduction. We wish to compute $u(x,t)$ at any point x and

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at any time, $t > 0$. We should make the following assumptions:

a) The rod is homogeneous, i. e. the mass of the rod per unit volume is constant, ρ (say)
b) The rod is insulated along its length.
c) The amount of heat crossing any section of the rod is given by $KS\left(\frac{\partial u}{\partial x}\right)\delta t$

Where, S = Area of cross section of the rod.

$\frac{\partial u}{\partial x} =$ Temperature gradient at the section, U

$\delta t =$ Time of flow of heat

$K =$ Thermal conductivity of the material of the rod

Now, the quantity of heat flowing into the element through U at $x = a$ in time δt is given by

$$= -KS\left(\frac{\partial u}{\partial x}\right)_x \delta t$$

where the negative sign has been taken because heat flows in the direction of decreasing temperature.

Again, the quantity of heat flowing through U at x=b in time δt

$$= -KS\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

Therefore, the amount of heat U obtains at time t is given by

$$= -KS\left(\frac{\partial u}{\partial x}\right)_x \delta t + KS\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

$$= KS\delta t\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad \text{------ (2.1)}$$

We assume that the above heat raises the temperature of the element by a small quantity δu . Then the same quantity of heat is given by

$$= (\rho S\delta x)c\delta u \quad \text{------ (2.2)}$$

where, c is the specific heat of the rod.

Since, the expressions (2.1) and (2.2) are equal, the we have

$$KS\delta t\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = (\rho S\delta x)c\delta u$$

$$\Rightarrow K\frac{u_x(x+\delta x, t) - u_x(x,t)}{\delta x} = \rho c\frac{\delta u}{\delta t}$$

------- (2.3)

As $\delta x \to 0$ and $\delta t \to 0$, equation (2.3) reduces to

$$K\frac{\partial^2 u}{\partial x^2} = \rho c\frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{---------(2.4)}$$

where, $k = \frac{K}{\rho c}$ is called the thermal diffusivity of the material of the rod. This equation is more popularly written with subscript notation as

$$u_t = ku_{xx} \quad (2.5)$$

We have now derived heat equation, also known as the diffusion equation.

3 Analytic Solution of the Heat Equation

In this section we present the analytic solution of the heat equation follows from Gerald [3], Smith [4] and Adam [9].

To find the analytic solution of the one dimensional heat equation we have to make it as an initial boundary problem (IBVP) by setting some initial conditions and boundary conditions.

If we have a rod of length $L = 1.5$ meter and $k = 0.02$. Setting the initial temperature distribution $u(x,0) = x$ and the boundary conditions defined with the condition that if one end of the rod w submerged in a liquid that is a constant 0^0 and the other end in a liquid at 120^0, then $u(0,t) = 0$ and $u(L,t) = u(1.5,\ t) = 120$ for all $t > 0$.

Finally, we arrive with the problem

$$\left.\begin{array}{l} u_t(x,t) = 0.02u_{xx}(x,t) \quad for\ 0 \le x \le 1.5 \\ and \quad t > 0 \\ u(x,0) = 0 \qquad for \quad 0 \le x \le 1.5 \\ u(0,t) = 0\ \ and\ \ u(1.5,\ t) = 120 \quad for\ t > 0 \end{array}\right\} \quad \text{------(3.1)}$$

Since the heat equation is linear, so we can find a linear combination of two solutions to equal another solution.

First of these solution is steady state solution $u_s(x)$, such that

$$u_s(x) = \left(\frac{T_L - T_0}{L}\right)x + T_0 \quad = 80x$$

The remaining part $u = v + u_s$ where $v = u - u_s$ .so it satisfies IBV problem,

$$\begin{array}{l} v_t(x,t) = kv_{xx}(x,t)\ ,\ for\ \ 0 \le x \le 1.5 \quad and \quad t > 0 \\ v(x,0) = u_0(x) - u_s(x) \quad for \quad 0 \le x \le L \\ v(0,t) = 0 = v(1,t) \quad for\ \ t > 0 \end{array} \quad \text{-------(3.2)}$$

By using separation of variable we will find the solution of v because separation of variables requires the problem homogeneous .Therefore we want to find the product solutions of the form $v(x,t) = X(x)T(t)$.

By separation of variable we get two ODE's for T and X. If we plug $v = X(x)T(t)$ in to the heat equation, we get $X(x)T(t) = KX''(x)T(t)$.

Dividing both sides by

$KT(t)$ *and* $X(x)$ gives, $\frac{T'(t)}{KT(t)} = \frac{X''(x)}{X(x)}$

Let μ be a constant such that $\frac{T'(t)}{KT(t)} = \mu$ and $\frac{X''(x)}{X(x)} = \mu$

Or, $T' + \mu KT = 0$ and $X'' - \mu x = 0$

Case I: If $\mu = 0$, the differential equation is $X'' = 0$ and has solution of general form, $X(x) = Ax + B$ with boundary conditions $0 = X(0) = B$ and $0 = X(L) = AL + B$.

Now we get , $A = B = 0$, again we arrive at a zero solution.

Case II: If $\mu = \lambda^2$ then $X'' - \lambda^2 x = 0$, $X = e^{\mu t}$.

$\therefore$ The general solution has the form $X(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$

and the boundary conditions are $0 = X(0) = C_1 + C_2$

$$0 = X(L) = C_1 e^{\lambda t} + C_2 e^{-\lambda t} \quad \text{------------(3.3)}$$

$\therefore C_2 = -C_1$

Therefore from (3.3) $0 = C_1(e^{\lambda L} + e^{-\lambda L})$

This implies that $C_1 = 0$ and $C_2 = 0$ which makes $X(x) = 0$.

Case III: If $\mu = -\lambda^2$ then $X'' + \lambda^2 x = 0$.

This has the general solution of the form $X(x) = A\cos\lambda x + B\sin\lambda x$

with the boundary conditions $X(0) = 0$ which means $A = 0$ and $X(L) = 0$ becomes $b\sin\lambda L = 0$.

Because $A = 0$ the cosine term disappears which means we need to solve $b\sin\lambda L = 0$.

Here, $\sin\lambda L = 0$, since $Sinx$ is equal to zero for positive integer values of π , $\sin\lambda L = 0$ will only happen when $\lambda L = n\pi$.

This leads to $\lambda^2 = \frac{n^2\pi^2}{L^2}$.

Therefore, $X(x) = b\sin\left(\frac{n\pi x}{L}\right)$.

Since we are looking for non-zero solutions we set $b = 1$,

$\therefore \mu_n = \frac{n^2\pi^2}{L^2}$ and $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

Here μ_n is an eigen-value of the Sturn-Liouville problem and $X_n(x)$ is an eigenfunction.

The complete solution to the Sturn-Liouville problem is the group of eigenvalues and eigenfunctions for $n = 1$ to $n = \infty$.

For the solutions to the heat equation, we have the product solution, $v_n(x,t) = e^{-\frac{n^2\pi^2 Kt}{L}} \sin\left(\frac{n\pi x}{L}\right)$

which satisfies the boundary conditions.

Again any linear combination of two solutions is also a solution. Therefore, we have

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 Kt}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad \text{----(3.4)}$$

which is also a solution to the heat equation. This solution satisfies all the conditions except the initial conditions, so we need to find the co-efficients b_n that satisfy the initial conditions. we use the initial condition to get , $v(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

For our problem, $-80x = \sum_{n=1}^{\infty} b_n Sin\frac{n\pi}{1.5}x$

This is known as the half range sine series expansion and b_n are calculated with

$$b_n = \frac{2}{L}\int_0^L v_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Expanding this gives

$$b_n = \frac{2}{1.5}\int_0^{1.5}(-80x)Sin\frac{n\pi}{1.5}x\,dx$$
$$= \frac{160}{n\pi}\left[(1.5Cosn\pi - 0) + \frac{1.5}{n\pi}(Sinn\pi - 0)\right]$$
$$= \frac{240}{n\pi}(-1)^n$$
$$\therefore\ b_n = (-1)^n\frac{240}{n\pi}$$

Plugging this function of into equation (3.4) gives a complete solution to the IBVP (3.2)

$$v(x,t) = \frac{240}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-0.02n^2\pi^2 t}\ Sin\frac{n\pi}{1.5}x$$

We conclude that the temperature in the rod is

$$u(x,t) = u_s(x) + v(x,t)$$

which gives

$$u(x,t) = 80x + \frac{240}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-0.02n^2\pi^2 t}\ Sin\frac{n\pi}{1.5}x \qquad \text{---------(3.5)}$$

which is the analytic solution to the IBVP (3.1).

4 Numerical Scheme for Heat Equation

As mentioned in section 2, the one dimensional heat equation with initial and boundary equation gives an initial boundary value problem (IBVP).

We investigate an efficient numerical scheme for the equation (2.5), follows from Gerald [3], Smith [4] and Burden [7].

In order to determine the scheme, we have to discretize the length and time. The discretization of $\frac{\partial u}{\partial t}$ is obtained by first order forward difference in time and the discretization of $\frac{\partial^2 u}{\partial x^2}$ is obtained by central difference in length.

The possible finite difference approximations for $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$:

Forward difference in time:
From Taylor's series we write

$$u(x,t+h) = u(x,t) + h\frac{\partial u(x,t)}{\partial t} + \frac{h^2}{2!}\frac{\partial^2 u(x,t)}{\partial t^2} + \ldots$$

$$\Rightarrow \frac{\partial u(x,t)}{\partial t} = \frac{u(x,t+h) - u(x,t)}{h} - o(h)$$

$$\therefore \frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+h) - u(x,t)}{h} \qquad \text{------(4.1)}$$

Central difference in space:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \partial^+\left(\frac{u(x,t) - u(x-k,t)}{k}\right)$$

$$\approx \frac{u(x+k,t) - u(x,t) - u(x,t) + u(x-k,t)}{k^2}$$

$$\approx \frac{u(x+k,t) - 2u(x,t) + u(x-k,t)}{k^2} \qquad ---(4.2)$$

We assume the uniform grid spacing with step size h and k for time and length respectively $t^{n+1} = t^n + h$ and $x_{i+1} = x_i + k$.

We also write u_i^n for $u(x,t)$ in equation (4.1) and (4.2).

Now equation (2.5) takes the form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t^{n+\frac{1}{2}}} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x_i^{\,2}}$$

$$\Rightarrow u_i^{n+1} = u_i^n + k\frac{\Delta t^{n+\frac{1}{2}}}{\Delta x_i^{\,2}}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

$$\Rightarrow u_i^{n+1} = (1-2\lambda)u_i^n + \lambda u_{i+1}^n + \lambda u_{i-1}^n \qquad \text{---(4.3)}$$

$$\text{where} \quad \lambda = k\frac{\Delta t^{n+\frac{1}{2}}}{\Delta x_i^{\,2}}$$

This is the explicit scheme for the equation (2.5).
Therefore, equation (4.3) leads the desired scheme for the heat equation.
The stencil for the explicit scheme (4.3) is presented below

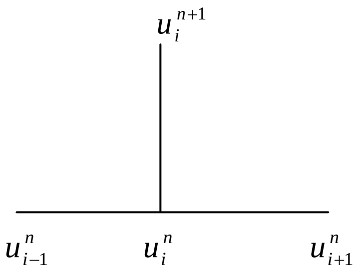

Fig 4.1: Stencil of the explicit scheme.

4.1 Stability Condition

Recalling the explicit finite difference scheme (4.3) for one dimension heat equation described the derivation in section 2

$$u_i^{n+1} = (1-2\lambda)u_i^n + \lambda u_{i+1}^n + \lambda u_{i-1}^n$$

The equation implies that if $\lambda \leq \frac{1}{2}$, the new solution is a convex combination of the three previous solutions that is the solution at new time-step $(n+1)$ at a spatial-node *i*, is an average of the solutions at the previous time-step at the spatial-nodes $i-1$, i and $i+1$.

Therefore the stability condition for the explicit scheme is

$$\lambda := \frac{k\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

which can be verified in the computer programming code very easily.

Finally, we have to choose Δt such that $\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$, where k is the thermal diffusivity of the material of the object (taken here $k = 0.22$).

5 Numerical Experiments and Results

To estimate the relative error between analytic solutions (3.5) stated in section 3 and numerical solution. We perform numerical experiments in $\Delta t = 0.022$ with step size $\Delta x = 0.15$ and the thermal diffusivity of the material of rod is k=0.36 which guarantees stability condition $\lambda = 0.1267 < \frac{1}{2}$. The relative error in estimated in L_1 norm defined by

$$\|e\|_1 := \frac{\|\rho_e - \rho_n\|_1}{\|\rho_e\|_1}$$ for all time where ρ_e is the exact solution (3.5) and ρ_n is the numerical solution computed by the finite difference scheme (4.3). Figure (5.1) represents the relative error. Fig 5.2 we present the initial temperature distribution of the rod.

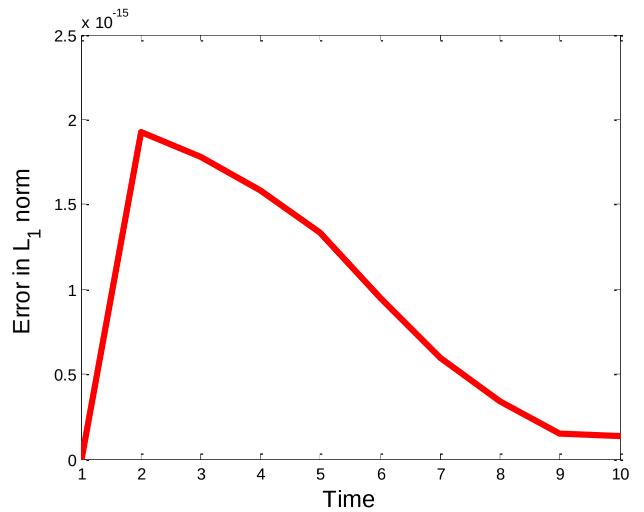

Fig 5.1: Relative Error.

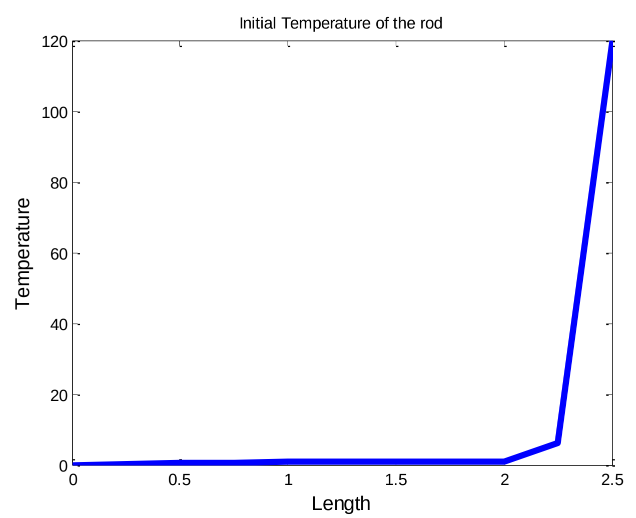

Fig 5.2: Initial Temperature distribution.

Fig 5.3 represents the temperature of initial position and after 10 minutes. We present the temperature profile for different time in Fig 5.4.

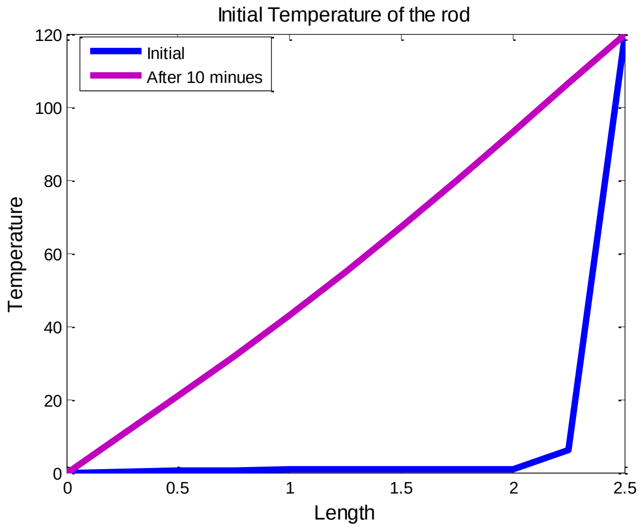

Fig 5.3: Initial temperature & temperature after 10 minutes.

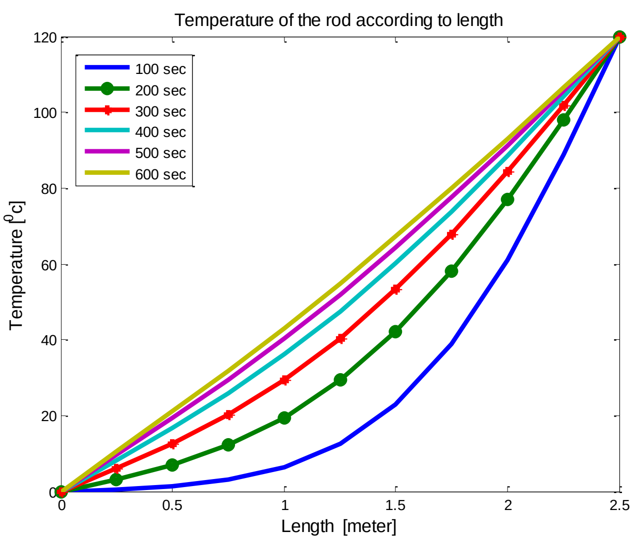

Fig 5.4: Temperature profile.

6 CONCLUSIONS

We have presented the derivation of a one dimensional heat equation and the analytic solution of heat equation as a BVP by the method of separation of variables. The derivation of explicit scheme for heat equation as an IBVP has been described and the stability condition of this scheme has also been presented. The relative error between the numerical solution and the analytic solution of heat equation has been computed using L_1 norm and the relative error is quite acceptable. We have presented temperature profiles for various time steps verifies well known qualitative behaviors of the heat equation.

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