Testing Correlation and Homoscedasticity in a Bivariate T-Population

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Abstract—Testing uncorrelation or independence of the two components of a bivariate normal distribution is known for independent and identically distributed bivariate normal observations. What happens if the some of three strong assumptions do not hold good? A recent trend among business experts and econometricians is to use bivariate *t*-distribution whose components are correlated and which has thicker sides. It has been pointed out that the tests for uncorrelation developed in the bivariate normal case remains the same for observations following identical bivariate *t*-distributions. The implication, in this case, is that the failure of rejection of hypothesis of uncorrelation of the two components on the basis of a test would not necessarily mean independence. Similarly, testing the equality of true variances or homoscedasticity by sample variance ratio is also well known for independent and identically distributed observations from two normal populations. What is less known is that even if the sample obervations follow independent and identical bivariate normal distributions, the test remains the same. In this paper, we prove that the test of equality of variances remains the same even for observations following identical bivariate *t*-distributions.

Keywords— correlation coefficient; variance ratio; bivariate t-distribution, homoscedasticity, test of hypotheses.

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1 INTRODUCTION

LET each of the sample observation $\underset{\sim}{X}_j,\ (j=1,2,\cdots,N)$, follow a bivariate *t*-distribution with mean $\underset{\sim}{\theta}$ (column vector order 2 components) and scale matrix Σ (a 2×2 matrix) where $\underset{\sim}{\theta}'=(\theta_1,\theta_2)$ and $\Sigma=(\sigma_{ik})$, $i=1,2;\ k=1,2$. We denote $\sigma_{11}=\sigma_1^2,\ \sigma_{22}=\sigma_2^2,\ \sigma_{12}=\rho\sigma_1\sigma_2$ with $\sigma_1>0,\ \sigma_2>0$ and the quantity ρ $(-1<\rho<1)$ as the product moment correlation coefficient between X_1 and X_2. The sample mean vector is $\underset{\sim}{\bar{X}}$, where $\underset{\sim}{\bar{X}}'=(\bar{X}_1,\bar{X}_2)$, so that the sums of squares and cross product matrix is given by $A=(a_{ik})$, where $a_{ik}=\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)(X_{kj}-\bar{X}_k)$, $i=1,2;k=1,2.$ Obviously, $a_{ii}=\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)^2$, $(i=1,2)$, and $a_{12}=\sum_{j=1}^{N}(X_{1j}-\bar{X}_1)(X_{2j}-\bar{X}_2)$. The sample correlation coefficient is then given by $r=a_{12}/(ms_1s_2)$, where $s_i^2=a_{ii}/m\ (i=1,2)$ and $m=N-1$.

In 1915, Fisher derived the distribution of the bivariate matrix A to study the distribution of correlation coefficient based on independent observations from a bivariate normal population [1]. The distribution of A is also known for uncorrelated observations following identical bivariate *t*-distributions. See, for example, in [2] page 157 and [3].

A recent interest among the applied scientists is the use of fat tailed distributions for modeling business data such as stock returns. Since the bivariate *t*-distribution has fatter tails, it has been increasingly applied for modeling business data. Interested readers may go through [4],[5] and [6].

If sample observations follow bivariate normal distributions, the test statistic $(m-1)^{1/2}(R-\rho)(1-R^2)^{-1/2}$ is known to have a *t*-distribution with $m-1$ degrees of freedom under the null hypothesis $H_0:\rho=0$ against the alternative $H_1:\rho\neq 0$. In this paper we will expound that the assumption of bivariate normality can be relaxed to bivariate *t*-distribution.

Consider the scaled variances $U=mS_1^2/\sigma_1^2$ and $V=mS_2^2/\sigma_2^2$. Assuming that the observations are from a bivariate normal population, Bose in 1935 and Finny in 1938 derived the density function of the variance ratio $H=U/V$ (See equation 4.1) [7],[8]. The distribution specializes to usual *F*-distribution $F(m,m)$ if $\rho=0$. The random variables U and V have a bivariate chi-square distribution [9] with correlation coefficient ρ^2 and found application in signal processing [10].

Can we relax the assumption of bivariate normality to fat tailed distributions, say, to bivariate *t*-distribution and study the behaviour of the variance ratio? In this paper, we prove that if the sample observations are uncorrelated *t*-distributions, the distribution of the variance ratio remains the same.

In Section 3, we derive the joint distribution of scaled variances (U, V) and correlation coefficient (R) whenever the sample observations are uncorrelated bivariate *t*-distributions. We pinpoint that the uncorrelation does not necessarily imply independence in this case. In Section 4, we prove that even if the observations follow identical bivariate *t*-distributions, the distribution of variance ratio $(H = U/V)$ is the same as that developed by Bose orFinney [7], [8].

2 Bivariate Normal and T-Distributions

Let $\underset{\sim}{X}$ be bivariate normal random vector with density function

$$f_{2.1}(\underset{\sim}{x}) = (2\pi)^{-1} |\Sigma|^{-1/2} \exp\left[\frac{-1}{2}\left((\underset{\sim}{x}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\theta})\right)\right], \quad (2.1)$$

which will be denoted by $N_2(\underset{\sim}{\theta}, \Sigma)$. Note that $E(\underset{\sim}{X}) = \underset{\sim}{\theta}$ and the variance-covariance matrix $Cov(\underset{\sim}{X}) = E(\underset{\sim}{X}-\underset{\sim}{\theta})(\underset{\sim}{X}-\underset{\sim}{\theta})'$ is given by $Cov(\underset{\sim}{X}) = \Sigma$. The bivariate normal distribution is also called elliptical normal distribution as the density function in (2.1) is constant on the ellipsoid $(\underset{\sim}{x}-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\theta}) = c^2$ for any constant c.

The j-th observation with i-th characteristic is denoted by x_{ij} so that the random sample is denoted by $\{X_{ij}; i=1,2; j=1,2,\cdots,N\}$. Then the sample $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \cdots \underset{\sim}{X}_N, (N>2)$ has the joint probability density function

$$f_{2.2}(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_N) = (2\pi)^{-N} |\Sigma|^{-N/2} \exp\left(\frac{-1}{2}\sum_{j=1}^{N}(\underset{\sim}{x}_j-\underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{x}_j-\underset{\sim}{\theta})\right). \quad (2.2)$$

Each observation $\underset{\sim}{X}_j (j=1,2,\cdots,N)$ in (2.2) follows (2.1). Since the observations in (2.2) are uncorrelated, by virtue of normality, they are also independent and we call it Independent and Identical Bivariate Normal (IIBN) model for sample. Note that sufficient statistics for $\underset{\sim}{\theta}$ and Σ exist in model (2.2).

In [1], Fisher derived the distribution of A in order to study the distribution of correlation coefficient from a normal sample. The distribution of A is given by

$$f_{2.3}(A) = 4\pi\Gamma(m-1)|\Sigma|^{-m/2}|A|^{(m-3)/2} \exp\left(-\frac{1}{2}tr\Sigma^{-1}A\right)$$
$$, A>0, m>2. \quad (2.3)$$

The joint density function of the elements of A can be written as

$$f_{24}(a_{11}, a_{22}, a_{12}) \alpha (\sigma_1\sigma_2)^{-m}(a_{11}a_{22} - a_{12}^2)^{(m-3)/2} \times$$
$$\exp\left(\frac{a_{11}}{2(1-p^2)\sigma_1^2} - \frac{a_{22}}{2(1-p^2)\sigma_2^2} + \frac{pa_{12}}{(1-p^2)\sigma_1\sigma_2}\right) \quad (2.4)$$

where,

$a_{11} > 0,\ a_{22} > 0,\ -\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{11}a_{22}},\ m > 2,\ -1 < \rho < 1.$

Let $\underset{\sim}{X}$ be bivariate t-random vector with probability density function

$$f_{2.5}(\underset{\sim}{x}) \propto |\Sigma|^{-1/2}\left(1+(\underset{\sim}{x}-\underset{\sim}{\theta})'(\nu\Sigma)^{-1}(\underset{\sim}{x}-\underset{\sim}{\theta})\right)^{-(\nu/2)-1}, \quad (2.5)$$

where, the scalar ν is assumed to be a known positive constant, see page 48 in [11]. The variable will be denoted by $T_2(\underset{\sim}{\theta}, \Sigma; \nu)$. Note that $E(\underset{\sim}{X}_j) = \underset{\sim}{\theta}$ and

$$Cov(\underset{\sim}{X}) = E(\underset{\sim}{X}-\underset{\sim}{\theta})(\underset{\sim}{X}-\underset{\sim}{\theta})' = \left(1-(2/\nu)\right)^{-1}\Sigma,\ \nu > 2.$$

Bivariate t-distribution can be generated by Conditionality Principle, Conditional Independence or by Stochastic Decomposition [12]. If sample observations $\underset{\sim}{X}_j (j=1,2,\cdots,N)$ are independent $T_2(\underset{\sim}{\theta}, \Sigma; \nu)$, the joint density for the sample is given by

$$f_{2.6}(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_N) \propto |\Sigma|^{-N/2}\prod_{j=1}^{N}\left(1+(\underset{\sim}{x}_j-\underset{\sim}{\theta})'(\nu\Sigma)^{-1}(\underset{\sim}{x}_j-\underset{\sim}{\theta})\right)^{-(\nu/2)-1}, \quad (2.6)$$

which will be called an Independent and Identical Bivariate T (IIBT) model for the sample. Note that sufficient statistics for θ and Σ do not exist in model (2.6). Is there any other alternative model for sample that shares intrinsic features, namely, marginality, conditionality, symmetry, equiprobable contour of (2.2)?

Now consider a sample $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \cdots \underset{\sim}{X}_N, (N>2)$ $X_1, X_2, \cdots X_N (N>2)$ having the joint probability density function

$$f_{2.7}(x_1, x_2, \ldots, x_N) \alpha |\Sigma|^{-N/2}$$
$$\left(1+\sum_{j=1}^{N}(x_j-\mu)'(\nu\Sigma)^{-1}(x_j-\mu)\right)^{-(\nu/2)-1} \quad (2.7)$$

Each observation $X_j (j=1,2,\cdots,N)$ in (2.7) follows (2.5). Since the observations in (2.7) are uncorrelated but not necessarily independent, (2.7) is called Uncorrelated and Identical Bivariate T (UIBT) model for the sample. Note that the sample observations in (2.7) are independent if $\nu \rightarrow \infty$ in which case (2.7) converges to (2.2) which is the joint density function of N uncorrelated observations from bivariate normal distribution. Samples can be generated from (2.7) by conditionality principle, conditional independence or much easily by stochastic decomposition [12]. In [13], Kelejian and Prucha proved that the Uncorrelated and Identical Bivariate model (2.7) captures

fat tailed behaviour better than the independent model in (2.6).
The density function of A based on UIBT model (2.7) is given by

$$f_{2.8}(A) \propto |\Sigma|^{-m/2} \quad |A|^{(m-3)/2}\left(1+tr(\nu\Sigma)^{-1}A\right)^{-(\nu/2)-m},$$
$$A>0, m>2 \qquad (2.8)$$

where, $m>2$, see page 160 in [3]. The above can also be written in terms of the elements as

$$f_{2.9}(a_{11},a_{22},a_{12}) \propto (\sigma_1\sigma_2)^{-m}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\left(1+\frac{1}{\nu(1-\rho^2)}\left(\frac{a_{11}}{\sigma_1^2}+\frac{a_{22}}{\sigma_2^2}-\frac{2\rho a_{12}}{\sigma_1\sigma_2}\right)\right)^{-(\nu/2)-m}, \qquad (2.9)$$

where, $a_{11}>0,\ a_{22}>0,\ -\sqrt{a_{11}a_{22}}<a_{12}<\sqrt{a_{11}a_{22}},\ m>2,\ -1<\rho<1.$

3 TESTS ON CORRELATION COEFFICIENT

3.1 Testing the Significance of Correlation Coefficient under Bivariate Normality

If each of the sample observations follow a bivariate normal distribution, it is well known that under the null hypothesis $H_0:\rho=0,$ the test statistic $R^2 \sim Beta(1/2,(m-1)/2)$, and $\sqrt{m-1}\,R\left(1-R^2\right)^{-1/2}$ has a Student *t*-distribution with $(m-1)$ degrees of freedom.The likelihood ratio test of the null hypothesis $H_0:\rho=0$ against the alternative $H_1:\rho\neq 0$ is done by the above statistic.

3.2 Testing the Significance of Correlation Coefficient under Bivariate t-Distribution

If sample observations follow UIBT model (2.7), then we need the distribution of R for testing the null hypothesis $H_0:\rho=0$ against the alternative $H_1:\rho\neq 0.$ It has been proved by Fang and Anderson; and Ali and Joarder in [2] and [14] respectively that the distribution of sample correlation coefficient remains the same as that for bivariate normal distribution. The proofs were done for a general class of distributions. For wide spectrum of readers, we sketch the proof in Theorem 3.2. The joint density function of scaled variances and correlation coefficient is derived below.

Theorem 3.1. *Let S_1^2, S_2^2 and R be sample variances and correlation coefficient based on a sample following bivariate Uncorrelated T-model (2.7). Then the joint density function of $U=mS_1^2/\sigma_1^2,\ V=mS_2^2/\sigma_2^2,\ R$ is given by*

$$f_{U,V,R}(u,v,r)\alpha(uv)^{(m/2)-1}$$
$$(1-r^2)^{(m-3)/2}\left(1+\frac{1}{\nu(1-p^2)}\left(u+v-2pr\sqrt{uv}\right)\right)^{-(\nu/2)-m} \qquad (3.1)$$

Where, $m>2$ and $-1<\rho<1.$

Proof. The density function of the elements of A based on UIBT model (2.7) is given by (2.9), where $a_{11}>0, a_{22}>0, -\infty<a_{12}<\infty, -1<\rho<1$, $m>2, \sigma_1>0, \sigma_2>0$. Under the transformation $a_{11}=ms_1^2, a_{22}=ms_2^2, a_{12}=mrs_1s_2$ (i.e., $a_{11}a_{22}=m^2s_1^2s_2^2,\ a_{12}^2=m^2r^2s_1^2s_2^2$) with Jacobian $J(a_{11},a_{22},a_{12}\rightarrow r,s_1^2,s_2^2)=m^3s_1s_2$, we have,

$$f_{S_1,S_2,R}(s_1,s_2,r)\alpha(s_1s_2)^{m-2}(1-r^2)^{(m-3)/2}$$
$$\left(1+\frac{1}{\nu(1-p^2)}\left(\frac{m^2s_1^2}{\sigma_1^2}+\frac{m^2s_2^2}{\sigma_2^2}\frac{2pms_1s_2}{\sigma_1\sigma_2}\right)\right)^{-(\nu/2)-m}$$

The transformation $ms_1^2=\sigma_1^2u,\ \ ms_2^2=\sigma_2^2v$ with Jacobian $J(s_1^2,s_2^2\rightarrow u,v)=m^{-2}\left(\sigma_1\sigma_2\right)^2$, leads to (3.1).

As $\nu\rightarrow\infty$, the joint density of U,V and R converges to the joint density function of U,V and R if the sample is drawn from a bivariate normal population [15].

Theorem 3.2. *Let S_1^2, S_2^2 and R be sample variances and correlation coefficient based on uncorrelated bivariate t-model (2.7). Then the density function of R is given by*

$$f_R(r)=\frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi\,\Gamma(m-1)}\left(1-r^2\right)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k!}\Gamma^2\left(\frac{m+k}{2}\right),$$
$$-1<r<1, \qquad (3.2)$$

where, $m>2$ and $-1<\rho<1.$

Proof. Since $u+v-2\rho r\sqrt{uv}\le \nu(1-\rho^2)$, by expanding the last term of (3.1), the probability density function of R can be written as

$$f_R(r)\alpha(1-r^2)^{(m-3)/2}$$
$$\int_0^{\alpha}\int_0^{\alpha}(uv)^{m/2-1}\left[1+\frac{1}{\nu(1-p^2)}\left(u+v-2pr\sqrt{uv}\right)\right]^{-(\nu/2)-m}dudv$$

Then the transformation $u=y_1\left(1-\rho^2\right),\ \mathrm{v}=y_2\left(1-\rho^2\right)$ with Jacobian $J(u,v\rightarrow y_1,y_2)=\left(1-\rho^2\right)^2$ yields

$$h_R(r) \propto \left(1-r^2\right)^{(m-3)/2} J(\rho r, m, \nu/2) \tag{3.3}$$

where,

$J(\rho,m,\nu)=\int_0^\infty\int_0^\infty (uv)^{m/2-1}\left(1+u+v-2\rho\sqrt{uv}\right)^{-\nu-m} dudv$. It can be evaluated that

$J(\rho,m,\nu)=\frac{\Gamma(\nu)}{\Gamma(m+\nu)}\sum_{k=0}^{\infty}\frac{(2\rho)^k}{k!}\Gamma^2\left(\frac{m+k}{2}\right)$, so that from (3.3), we have

$h(r)\propto\left(1-r^2\right)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho)^k}{k!}\Gamma^2\left(\frac{m+k}{2}\right)$, which, apart from normalizing constant, is known to be the density function of R.

The density function of R was derived originally by Fisher in 1915 for independent sample observations following identical bivariate normal distributions [1]. Theorem 3.2 indicates that the assumption of bivariate normality under which tests on correlation coefficient are developed can be relaxed to bivariate *t*-distribution.

Acceptance of the null hypothesis does not mean independence unless the sample is from bivariate normal distribution. In view of Theorem 3.2, the test is true for bivariate *t*-distribution in which case acceptance of $H_0:\rho=0$ implies uncorrelation but not necessarily independence.

We warn that the distribution of R is not necessarily the same if we had independent model (2.6) for the sample. The approximate distribution of R for independent observations from bivariate *t*-population can be obtained from page 157 in [11]. For the distributions of R in nonelliptical populations, the reader is referred to [16] and the references therein.

4 Testing Equality of Variances

Consider testing the equality of variances under three different situations. Suppose that we want to test the hypothesis $H_0:\sigma_1^2=\sigma_2^2$, against the alternative hypotheses $H_0:\sigma_1^2\neq\sigma_2^2$.

4.1 Independently and Identically Distributed Observations from $N(\theta_1,\sigma_1^2)$ and $N(\theta_2,\sigma_2^2)$

Consider two independent samples $X_{1j}\ (j=1,2,\cdots,N)$ and $X_{2j}\ (j=1,2,\cdots,N)$ from $N(\theta_1,\sigma_1^2)$ and $N(\theta_2,\sigma_2^2)$ respectively. Then the likelihood ratio is given by

$\lambda=\frac{2^m H^{m/2}}{(1+H)^m}$, Where, $H=U/V$. Since, λ is a monotonic function of H, the test can be carried out by H. The critical region $\{\lambda:0<\lambda<\lambda_0\}$ is equivalent to $\{H:\{H\le H_l\}\cup\{H\ge H_u\}\}$ where H_u and H_l can be determined so that under null hypothesis $P(H\ge H_u)=\alpha/2$ and $P(H\ge H_l)=1-(\alpha/2)$.

4.2 Independently and Identically Distributed Observations from $N_2(\underset{\sim}{\theta},\Sigma)$

Consider observations $X_1,X_2,\cdots X_N\ (N>2)$ each having $N_2(\underset{\sim}{\theta},\Sigma)$. Then it follows from Bose (1935) or Finney (1938) that H has a density function

$$f_H(h)=\frac{(1-\rho^2)^{m/2}}{B\left(\frac{m}{2},\frac{m}{2}\right)}\frac{h^{(m-2)/2}}{(1+h)^m}\left(1-\frac{4\rho^2 h}{(1+h)^2}\right)^{-(m+1)/2},\ h>0,$$

Which, will be denoted by $F(m,m;\rho)$. In [8], Finney compared the variability of the measurements of standing height and stem length for different age group of school boys by his method with the help of Hirschfeld [17]. In [18], Wilks developed the likelihood ratio test for testing the equality of variances in presence of correlation if the parent population is bivariate normal. An excellent review is available in [19] by Modarres, who also performed Monte Carlo simulation to determine the behavior of the likelihood ratio test.

4.3 Uncorrelated and Identically Distributed Observations from Bivariate T-Distribution

In this section we will prove that the even if each of the sample observations $\underset{\sim}{X}_1,\underset{\sim}{X}_2,\cdots\underset{\sim}{X}_N\ (N>2)$ follow identical bivariate *t*-distribution and the sample has the model (2.7), the distribution of $H=U/V$ remains the same as $F(m,m;\rho)$. Note that the observations are uncorrelated and not necessarily independent though the correlation between the components X_{1j} and X_{2j} of $X_j(j=1,2,\cdots,N)$ is ρ [14].

Theorem 4.1 Let S_1^2,S_2^2 and R be sample variances and correlation coefficient based on uncorrelated bivariate *t*-model (2.7). Also let $U=mS_1^2/\sigma_1^2$ and $V=mS_2^2/\sigma_2^2$ be scaled sample variances. Then the density function of $H=U/V$ is given by

$$f_H(h)=\frac{(1-\rho^2)^{m/2}}{B\left(\frac{m}{2},\frac{m}{2}\right)}\frac{h^{(m-2)/2}}{(1+h)^m}\left(1-\frac{4\rho^2 h}{(1+h)^2}\right)^{-(m+1)/2},\ h>0, \tag{4.1}$$

Where, $m>2$ and $-1<\rho<1$, and $B(a,b)$ is the usual beta function.

Proof. It follows from (3.1) that the joint density function of U and V is given by

$$f_{U,V}(u,v)\alpha(uv)^{(m/2)-1}\int_{-1}^{1}(1-r^2)^{(m-3)/2}\left(1+\frac{1}{v(1-p^2)}\left(u+v-2pr\sqrt{uv}\right)\right)^{-(v/2)-m}dr \qquad (4.2)$$

It follows that the density function of $H=U/V$ is given by

$$f_H(h)\alpha h^{(m-2)/2}\int_{r=-1}^{1}\left(1-r^2\right)^{(m-3)/2}\int_{v=0}^{\alpha}v^{m-1}\left(1+\frac{1}{v(1-p^2)}\left(vh+v-2pr\sqrt{vhv}\right)\right)^{-(v/2)-m}dvdr$$

Substituting $v(1+h)-2\rho rv\sqrt{h}=y(1-\rho^2)$, with the Jacobian $J(v\to y)=\dfrac{1-\rho^2}{(1+h)-2\rho r\sqrt{h}}$, we have

$$f_H(h)\alpha\frac{h^{(m-2)/2}}{(1+h)^m}\int_{r=-1}^{1}\left(1-\frac{2pr\sqrt{h}}{1+h}\right)^{-m}\int_{v=0}^{\infty}v^{m-1}(1+(y/v))^{-(v/2)-m}dydr \qquad (4.3)$$

Since the integral in y can be converted to a beta type integral and it gets absorbed into the normalizing constant,

$$f_H(h)\propto\frac{h^{(m-2)/2}}{(1+h)^m}\int_{r=-1}^{1}\left(1-\frac{2\rho r\sqrt{h}}{1+h}\right)^{-m}dr. \qquad (4.4)$$

which is equivalent to what was obtained by Bose in 1935 [7] or Finney in 1938 [8], and is equivalent to (4.1).

Equation (4.1) is well known for bivariate normal distribution (Bose, 1935). This proves the robustness of variance ratio in the class of bivariate elliptical *t*-distributions. The distribution of test statistic $H=U/V$ given by (4.1) will be denoted by $F(m,m;\rho)$.

Example 4.1 A chemical engineering is investigating the effect of process operating temperature (x) on product yield (y). The study results in the following data:

x	y
100	45
110	51
120	54
130	61
140	66
150	70
160	74
170	78
180	85
190	89

See page 457 in [20]. We assume that X and Y have an elliptical *t*-distribution with correlation coefficient ρ given by (2.7).

a. We want to test $H_0:\rho=0$ against $H_1:\rho\neq 0$. The statistic $T=\dfrac{R\sqrt{m-1}}{\sqrt{1-R^2}}$ has a *t*-distribution with a degrees of freedom of $m-2$ so that the Rejection Region is $\{t:-2.306<t<2.306\}$. Since the sample produces

$$r=\frac{a_{12}}{\sqrt{a_{11}a_{22}}}=\frac{3985}{\sqrt{(8250)(1932.10)}}=0.998128718$$

and

$t=\dfrac{r\sqrt{m-1}}{\sqrt{1-r^2}}\approx 48.9696,$ we reject the null hypothesis and accept the alternative hypothesis. In the classical method, the assumption of bivariate normality of X and Y and the independence of bivariate observations were required.

b. We want to test $H_0:\sigma_1^2=\sigma_2^2$ against $H_1:\sigma_1^2<\sigma_2^2$ where σ_1^2 and σ_2^2 are the true variations of temperature and that of product yield. Let s_1^2 and s_2^2 are the sample variations of temperature and that of product yield.

The statistic $F=\dfrac{s_1^2/m}{s_2^2/m}$, has a variance ratio distribution with the same degrees of freedom of freedom of 9 so that the Rejection Region is $\{F:0.3145<F<3.18\}$. Since the sample yields $F=\dfrac{1932.10/9}{8250/9}\approx 0.2342$, we do not reject null hypotheses. In the classicalmethod, the assumption of normality of X and Y and their independence are required.

5 CONCLUSION

The testing of equality of variances in presence of correlation with a bivariate normal population has a long history. Under the null hypothesis, the test statistic $H=U/V$ has a $F(m,m;\rho)$ distribution and can be used for testing $H_0:\sigma_1^2=\sigma_2^2,\ \rho\neq 0$ against $H_0:\sigma_1^2\neq\sigma_2^2,\ \rho\neq 0$ if ρ is known.

In this paper, we have proved that the assumption of normality can be relaxed to bivariate *t*- distribution for testing equality of variances in presence of correlation. However, the acceptance of the null hypothesis in this case would mean uncorrelation; It would mean independence in the special case of bivariate normality. The robustness of the distribution of the variance ratio or of the test will stimulate statisticians, econometricians and business experts to embark on further investigation in the area, let alone the use of classical results with confidence.

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