Numerical Study on the Stability of Finite Difference Schemes for Solving Advection Diffusion Equation

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Abstract

This paper presents a numerical study with two explicit finite difference schemes FTBSCS (forward time backward space and centered space) and FTCS (forward time and centered space) for solving the advection diffusion equation (ADE). Stability conditions for the schemes are studied, and numerical experiments are performed by applying the stability criterions obtained in this study. Error comparisons with analytical solutions of ADE are presented graphically to show the accuracy of the schemes.

Keywords—Advection Diffusion Equation, Explicit Finite Difference Schemes, Stability condition.

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1 INTRODUCTION

The most general statement of conservation of contaminant mass in a control volume subject to advective and diffusive flux across its boundaries is $\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}$, where c(x,t)=solute concentration, D=solute diffusivity, and u= mean flow velocity assumed to be constant. The equation assumes incompressible ambient fluid, and adopts Fick's law of simple proportionality between diffusive contaminant and the concentration gradient. It is a parabolic type partial differential equation, and is derived on the principle of conservation of mass using Fick's law. This equation reflects physical phenomena where in the diffusion process particles are moving with certain velocity from higher concentration to lower concentration. This process is described by the right hand term of the Advection diffusion equation. Second and right hand terms represent the concentration of the contaminant particles as respect to the change in distance and the acceleration in velocity gained over distance, respectively. Since stability results for many common schemes for approximating, the wave equation and the heat equation are well known, an often used practical strategy is to take the more restrictive of the two stability constraints for the wave and heat equations as the stability condition for the advection diffusion equation. The definition of stability that we employ here is a generalization of the classical Neumann stability conditions and is designed to guarantee that the computed solution inherits one important property of the analytical solution. In this paper, we are interested in the numerical study on the stability criteria of approximation schemes for solving this equation.

Many researchers have already been worked on it. Derivation of ADE, analytical solution, and numerical simulation of ADE are studied in the literatures [1], [2], and papers [10-12], [17]. Atul Kumar, Dilip Kumar Jaiswaland Naveen Kumar [5] presented an analytical solution of one dimensional advection diffusion equation with variable coefficients in a finite domain by using Lap lace transformation technique. In that process new independent space and time variables have been introduced. Ogata and Banks [6] obtained analytical solution of the one dimensional ADE by reducing the original ADE into a diffusion equation by applying moving coordinates. F.B. Agusto and O. M. Bamingbola [7] obtained a Numerical Treatment of the Mathematical Models for Water Pollution. Young-San Park, Jong-Jin Baik [8] presented an analytical solution of the advection diffusion equation for a ground level finite area source. Al-Niami and Ruston, 1977 [9] obtained analytical solution of the one dimensional ADE by reducing the original ADE into a diffusion equation by introducing another dependent variable. L.F. Leon, P.M. Austria [13] presented Stability Criterion for Explicit Scheme on the solution of Advection-Diffusion Equation. T. F. Chan [14] presented Stability analysis of finite difference schemes for the advection diffusion equation. Alain Rigal [15] obtained Stability analysis of finite difference schemes for the Navier-Stokes equations. K.W. Morton [16] obtained Stability and convergence in fluid flow problems.

With the above discussion in view, in the present paper, Numerical Study on the Stability of Finite Difference Schemes with FTBSCS and FTCS techniques for solving the advection diffusion equation is presented. Numerical experiments are performed to verify the stability criterions obtained in this study. The schemes are compared with an analytical solution of ADE graphically to show accuracy of the solutions.

2 Numerical Schemes for Governing equation

We consider the one–dimensional ADE as an initial and boundary value problem $\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}$ with initial condition $c(t_0,x)=c_0(x);\ \ a\le x\le b$ and boundary conditions $c(t,a)=c_a(x)\,;\quad t_0\le t\le T\ \ c(t,b)=c_b(x)\,;\quad t_0\le t\le T$. Finite difference techniques for solving the one dimensional advection diffusion equation can be considered according to the number of spatial grid points involved, the number of time levels used, whether they are explicit or implicit nature.

2.1 Explicit finite difference scheme for ADE

For the numerical solution of the one –dimensional linear advection- diffusion equation we consider the IBVP $\frac{\partial \mathrm{C}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{C}}{\partial \mathrm{x}}=\mathrm{D}\frac{\partial^2 \mathrm{C}}{\partial \mathrm{x}^2}$……………………. (1) With initial condition $\mathrm{c(x,0)=f(x)}\quad 0\le \mathrm{x}<\mathrm{l}$ and Neumann boundary conditions $\frac{\partial \mathrm{c}}{\partial \mathrm{x}}(\mathrm{x}=0,\mathrm{t})=0;\quad 0<\mathrm{t}\le \mathrm{T}$ and $\frac{\partial \mathrm{c}}{\partial \mathrm{x}}(\mathrm{x}=\mathrm{l},\mathrm{t})=0;\quad 0<\mathrm{t}\le\mathrm{T}$. In order to develop the scheme, we discretize the x-t plane by choosing a mesh width h≡Δx space size and a time step size k≡Δt. The finite difference methods, we will develop, produce approximations $\mathrm{c_i^n}\in \mathrm{R^n}$ to the solution $\mathrm{c(x_i,t_n)}$ in the discrete points by $\mathrm{x_i}$= ih, i= 0, 1, 2, 3, … … … and $\mathrm{t_n}$= nk, n= 0, 1, 2, 3, … … …Let the solution $\mathrm{c(x_i,t_n)}$ be denoted by $\mathrm{c_i^n}$ and its approximate value by $\mathrm{c_i^n}$.

2.2 Explicit Centered Difference Scheme by FTBSCS Techniques

By Explicit forward time difference formula $\frac{\partial c}{\partial t}=\frac{c_i^{n+1}-c_i^n}{\Delta t}$ … … … (2), next use the backward space difference formula $\frac{\partial c}{\partial x}=\frac{c_i^n-c_{i-1}^n}{\Delta x}$ … … … (3) and centered space difference formula $\frac{\partial^2 c}{\partial x^2}=\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{\Delta x^2}$ … … … (4). Substituting equations (2- 4) into equation (1) and rearrange according the time level, lead to $\frac{c_i^{n+1}-c_i^n}{\Delta t}+u\frac{c_i^n-c_{i-1}^n}{\Delta x}=D\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{\Delta x^2}$

Which leads to $c_i^{n+1}=c_i^n-\frac{u\Delta t}{\Delta x}(c_i^n-c_{i-1}^n)+\frac{D\Delta t}{\Delta x^2}(c_{i+1}^n-2c_i^n+c_{i-1}^n)$

$$c_i^{n+1}=\left(\frac{u\Delta t}{\Delta x}+\frac{D\Delta t}{\Delta x^2}\right)c_{i-1}^n+\left(1-\frac{u\Delta t}{\Delta x}-2\frac{D\Delta t}{\Delta x^2}\right)c_i^n+\frac{D\Delta t}{\Delta x^2}c_{i+1}^n$$

Implies to $c_i^{n+1}=(\gamma+\lambda)c_{i-1}^n+(1-\gamma-2\lambda)c_i^n+\ \lambda c_{i+1}^n$ … … … (5) in which, $\gamma=\frac{u\Delta t}{\Delta x}$, $\lambda=\frac{D\Delta t}{\Delta x^2}$

2.3 Explicit Centered Difference Scheme by FTCS techniques

By Explicit forward time difference formula $\frac{\partial c}{\partial t}=\frac{c_i^{n+1}-c_i^n}{\Delta t}$ … … … (6), next use the centered space difference formula $\frac{\partial c}{\partial x}=\frac{c_{i+1}^n-c_{i-1}^n}{2\Delta x}$ … … … (7) and $\frac{\partial^2 c}{\partial x^2}=\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{\Delta x^2}$ … … … (8)

Substituting equations (6 - 8) into equation (1) and rearrange according the time level, lead to

$$\frac{c_i^{n+1}-c_i^n}{\Delta t}+u\frac{c_{i+1}^n-c_{i-1}^n}{2\Delta x}=D\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{\Delta x^2}$$

Which leads to $c_i^{n+1}=c_i^n-\frac{u\Delta t}{2\Delta x}(c_{i+1}^n-c_{i-1}^n)+\frac{D\Delta t}{\Delta x^2}(c_{i+1}^n-2c_i^n+c_{i-1}^n)$

$$c_i^{n+1}=\left(\frac{u\Delta t}{2\Delta x}+\frac{D\Delta t}{\Delta x^2}\right)c_{i-1}^n+\left(1-2\frac{D\Delta t}{\Delta x^2}\right)c_i^n+\left(\frac{D\Delta t}{\Delta x^2}-\frac{u\Delta t}{2\Delta x}\right)c_{i+1}^n$$

Implies to

$c_i^{n+1}=(\gamma/2+\lambda)c_{i-1}^n+(1-2\lambda)c_i^n+\left(-\frac{\gamma}{2}+\lambda\right)c_{i+1}^n$ … (9), in which, $\gamma=\frac{u\Delta t}{\Delta x}$, $\lambda=\frac{D\Delta t}{\Delta x^2}$

Now, we can write the general form of an explicit 1st order scheme as $c_i^{n+1}=L_0c_{i-1}^n+L_1c_i^n+L_2c_{i+1}^n$

where, the values of the coefficients are given at below:

Coefficient of explicit central difference scheme			
Scheme	L_0	L_1	L_2
FTBSCS	$\gamma+\lambda$	$1-\gamma-2\lambda$	λ
FTCS	$\frac{\gamma}{2}+\lambda$	$1-2\lambda$	$\lambda-\frac{\gamma}{2}$

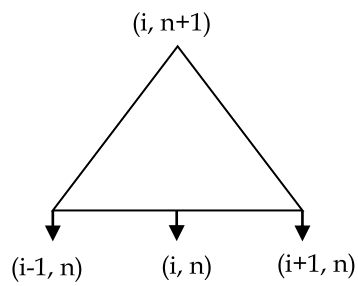

Figure 2.3: The stencil for explicit method for the ADE.

So, from above equations, we observe that knowing the values at time n we can obtain the corresponding ones at time n+1 using this recurrence relation c_0^n and c_i^nmust be replaced by the boundary conditions, in this example they are both 0 and 1.

3. Stability Analysis

After surveying the relevant literature on the subject, we discover that no practical stability criterion exists for (5) and (9). We developed simultaneous stability conditions for both the schemes and maintaining the criteria we verify the results of the schemes by setting an example.

3.1 Stability conditions for the scheme with FTBSCS techniques are given by

$$0\le\frac{D\Delta t}{\Delta x^2}\le 1 \text{ and } -\frac{D\Delta t}{\Delta x^2}\le\frac{u\Delta t}{\Delta x}\le 1-2\frac{D\Delta t}{\Delta x^2}.$$

In which $\gamma=\frac{u\Delta t}{\Delta x}$, $\lambda=\frac{D\Delta t}{\Delta x^2}$

3.2 Stability conditions for the scheme with FTCS techniques are given by

$$0\le\frac{D\Delta t}{\Delta x^2}\le\frac{1}{2} \text{ and } -2\frac{D\Delta t}{\Delta x^2}\le\frac{u\Delta t}{\Delta x}\le 2\left(1-\frac{D\Delta t}{\Delta x^2}\right),$$

In which, $\gamma=\frac{u\Delta t}{\Delta x}$, $\lambda=\frac{D\Delta t}{\Delta x^2}$

4. Numerical Simulation and results discussions

Various finite difference equations were used to represent the parabolic model equation (1). It is extremely important to experiment with the application of these numerical techniques. It is hoped that by writing computer codes and analyzing the results, additional insights into the solution procedures are gained. Therefore, this section proposes an example and presents solutions by the described schemes.

4.1 Problem description: Estimation of pollutant in a river of length l = 6 meter at all time t = 1minute to t = 6 minutes with fluid velocity, u = 0.01 m/s = 36 m/h and diffusion coefficient, D = 0.01m2/s=36 m2/h. The advection diffusion equation for this problem is $\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}$. Various values of spatial nodes size and time steps are to be used to investigate the numerical schemes and the effect of steps on stability and accuracy. An attempt is made to solve the stated problem subject to the imposed initial and Neumann boundary conditions by the following: The FTBSCS and FTCS schemes with

$\Delta x = 0.05$, nx = 120 $\Delta t = 0.07$, nt = 3600

$\Delta x = 0.15$, nx = 40 $\Delta t = 0.1$, nt = 3600

$\Delta x = 0.05$, nx = 120 $\Delta t = 0.122$, nt= 2948

Solutions

Case I. When the step sizes are $\Delta x = 0.05$, $\Delta t = 0.07$.

In this case, both the schemes are to be used as stated previously: The stability requirements of the FTBSCS scheme are $0 \le \frac{D\Delta t}{\Delta x^2} \le 1$ and $-\frac{D\Delta t}{\Delta x^2} \le \frac{u\Delta t}{\Delta x} \le 1 - 2\frac{D\Delta t}{\Delta x^2}$ (the terms $\frac{u\Delta t}{\Delta x} = \gamma$ and $\frac{D\Delta t}{\Delta x^2} = \lambda$ are known as the advection number and diffusion number respectively).

For this particular application, $\lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.07}{(0.05)^2} = 0.28$

$$\gamma = \frac{u\Delta t}{\Delta x} = \frac{0.01\times0.07}{0.05} = 0.014$$

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.07}{(0.05)^2} = 0.28 \le 1 \text{ and } -\frac{0.01\times0.07}{(0.05)^2} \le \frac{0.01\times0.07}{0.05} \le 1 - 2\times\frac{0.01\times0.07}{(0.05)^2}$$

or,

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.07}{(0.05)^2} = 0.28 \le 1 \text{ and } -0.28 \le 0.014 \le 0.44.$$

And the stability requirements of the FTCS scheme are $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$ *and* $-2\frac{D\Delta t}{\Delta x^2} \le \frac{u\Delta t}{\Delta x} \le 2\left(1 - \frac{D\Delta t}{\Delta x^2}\right)$.

For this particular application, $\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.07}{(0.05)^2} = 0.28 \le \frac{1}{2}$ and $-2\times\frac{0.01\times0.07}{(0.05)^2} \le \frac{0.01\times0.07}{0.05} \le 2\left(1 - \frac{0.01\times0.07}{(0.05)^2}\right)$

or,

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.07}{(0.05)^2} = 0.28 \le \frac{1}{2} \text{ and } -0.56 \le 0.014 \le 1.44.$$

Therefore, the stability conditions for both the schemes are satisfied, and a stable solution is expected. The velocity profiles are to be obtained up to t = 4 minutes are shown in Figure 4.1.

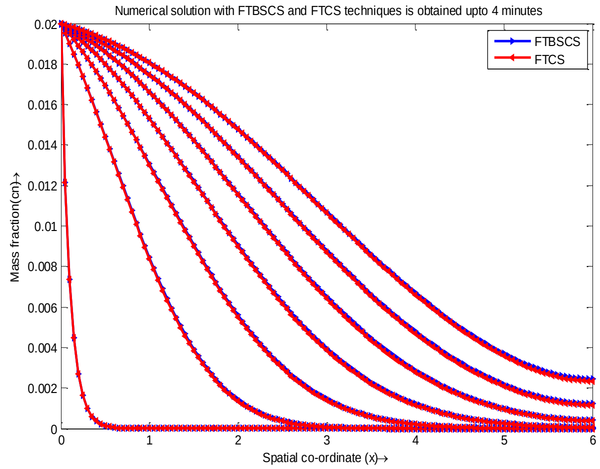

Figure 4.1: Velocity profiles with the schemes, Δx=0.05, Δt=0.07

Case II. When the step sizes are increased to Δx = 0.15, Δt = 0.1, the stability requirements of the FTBSCS scheme are $0 \leq \frac{D\Delta t}{\Delta x^2} \leq 1$ and $-\frac{D\Delta t}{\Delta x^2} \leq \frac{u\Delta t}{\Delta x} \leq 1 - 2\frac{D\Delta t}{\Delta x^2}$. For this particular application, $\lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.1}{(0.15)^2} = 0.044$ and $\gamma = \frac{u\Delta t}{\Delta x} = \frac{0.01\times0.1}{0.15} = 0.007$

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.1}{(0.15)^2} = 0.044 \leq 1 \text{ and } -\frac{0.01\times0.1}{(0.15)^2} \leq \frac{0.01\times0.1}{0.15} \leq 1 - 2\times\frac{0.01\times0.1}{(0.15)^2}$$

or

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.1}{(0.15)^2} = 0.044 \leq 1 \text{ and } -0.044 \leq 0.007 \leq 0.912$$

And the stability requirements of the FTCS scheme are $0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$ and $-2\frac{D\Delta t}{\Delta x^2} \leq \frac{u\Delta t}{\Delta x} \leq 2\left(1 - \frac{D\Delta t}{\Delta x^2}\right)$.

For this particular application,

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.1}{(0.15)^2} = 0.044 \leq \frac{1}{2} \text{ and } -2\times\frac{0.01\times0.1}{(0.15)^2} \leq \frac{0.01\times0.1}{0.15} \leq 2\left(1 - \frac{0.01\times0.1}{(0.15)^2}\right)$$

or

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times0.1}{(0.15)^2} = 0.044 \leq \frac{1}{2} \text{ and } -0.088 \leq 0.007 \leq 1.912$$

Therefore, the stability condition is satisfied, and a stable solution is expected. The velocity profiles are to be obtained up to t = 6 minutes are shown in Figure 4.2.

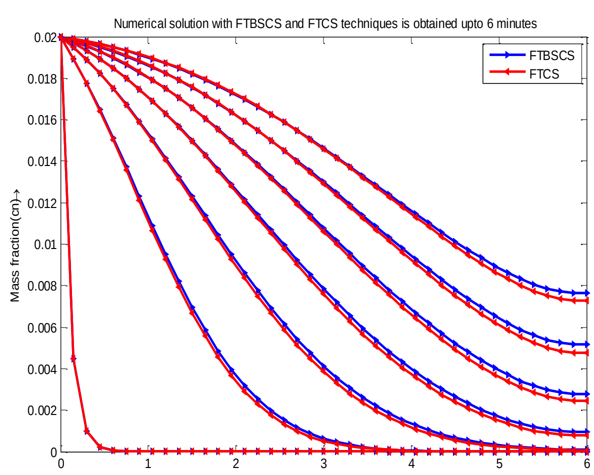

Figure 4.2: Velocity profiles with the schemes, Δx=0.15, Δt=0.1

Case III. When the step sizes are increased to Δx = 0.05, Δt = 0.122, which is only a fraction of an increase over preceding cases. In this case, the stability requirement of the FTBSCS scheme are $0 \le \frac{D\Delta t}{\Delta x^2} \le 1$ and $-\frac{D\Delta t}{\Delta x^2} \le \frac{u\Delta t}{\Delta x} \le 1 - 2\frac{D\Delta t}{\Delta x^2}$.

For this particular application, $\lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.122}{(0.05)^2} = 0.488$ and $\gamma = \frac{u\Delta t}{\Delta x} = \frac{0.01\times 0.122}{0.05} = 0.0244$

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.122}{(0.05)^2} = 0.488 \le 1 \text{ and } -\frac{0.01\times 0.122}{(0.05)^2} \le \frac{0.01\times 0.122}{0.05} \le 1 - 2\times\frac{0.01\times 0.122}{(0.05)^2}$$

or

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.122}{(0.05)^2} = 0.488 \le 1 \text{ and } -0.488 \le 0.0244 \le 0.024,$$ which exceeds the stability requirement.

And the stability requirements of the FTCS scheme are $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$ and $-2\frac{D\Delta t}{\Delta x^2} \le \frac{u\Delta t}{\Delta x} \le 2\left(1 - \frac{D\Delta t}{\Delta x^2}\right)$.

For this particular application,

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.122}{(0.05)^2} = 0.488 \le \frac{1}{2} \text{ and } -2\times\frac{0.01\times 0.122}{(0.05)^2} \le \frac{0.01\times 0.122}{0.05} \le 2\left(1 - \frac{0.01\times 0.122}{(0.05)^2}\right)$$

or

$$\frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.122}{(0.05)^2} = 0.488 \le \frac{1}{2} \text{ and } -0.976 \le 0.0244 \le 1.024.$$

Therefore, at this stage one of the stability conditions for FTBSCS is not satisfied, and an unstable solution is appeared. With the step sizes indicated, an unstable solution is developed. The velocity profiles are to be obtained at t = 6 minutes are shown in Figure 4.3.

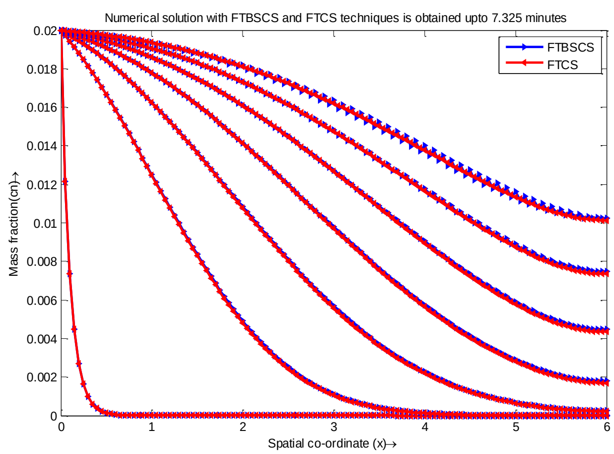

Figure 4.3: Velocity profiles with the schemes, Δx=0.05, Δt = 0.122

5. Analysis

In the preceding section, two explicit finite difference schemes are applied to the advection diffusion equation and the solutions are presented. The effect of the stability imposed by the diffusion number on the FTBSCS and FTCS explicit schemes are clearly indicated. Therefore, for these schemes the selection of step sizes is limited due to the stability requirement. However, the accuracy requirement limits the use of large time steps, since an increase in time steps will increase the truncation errors introduced in the approximation process of the PDE.

For the simple problem under consideration, an analytical solution may be obtained. The analytical solution of ADE with the imposed initial and boundary conditions is as follows-

5.1 The Advection-diffusion equation as an IBVP

The one-dimensional advection-diffusion equation [1] is given as $\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$ ... ... (10) where, c represents the solute concentration [ML-3] at x along longitudinal direction at time t, D is the solute dispersion, if it is independent of position and time, is called dispersion coefficient [L2T-1], t=time[T]; x= distance[L] and, u is the mean flow velocity [LT-1] assumed to be constant.

Appended with initial condition $c(x,0) = f(x) \qquad 0 \le x < l$ ……… (11)

and boundary conditions $c(x=0,t) = g_0(x) \qquad 0 < t \le T$ ……… (12)

$c(x=l,t) = g_1(x) \qquad 0 < t \le T$ ……… (13)

the ADE formulates an initial boundary value problem (IBVP).

5.2 Analytic solution

By coordinate transformation, the exact solution [1] of the advection-diffusion equation in unbounded is given by

$$c(x,t) = \frac{M}{A\sqrt{4\pi D t}} exp\left(-\frac{(x - (x_0 + ut))^2}{4Dt}\right) \text{ ………… } (14)$$

5.3 Error Estimation and Convergence

An error term is in L1-norm as $err = \frac{\|c_e - c_n\|_1}{\|c_n\|}$. ……… (15) where, ce is the exact solution, and cn is the numerical solution computed by the finite difference schemes for time $t \in [0,6]$.The following figure 5.1 shows the convergence of relative error by the scheme FTBSCS.

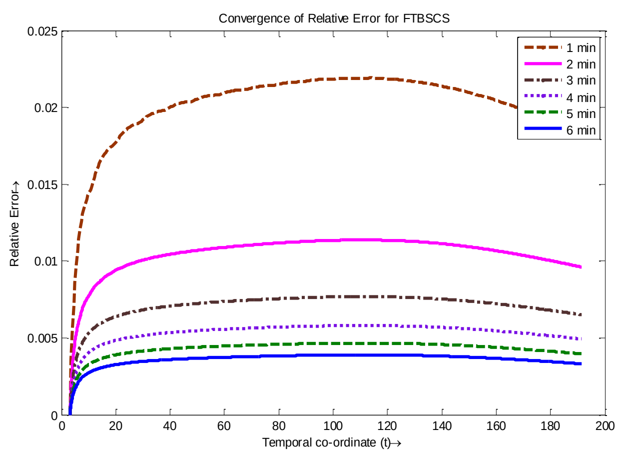

Figure 5.1 Rate of Numerical feature of Convergence

The following figure 5.2 shows the convergence of relative error by the scheme FTCS.

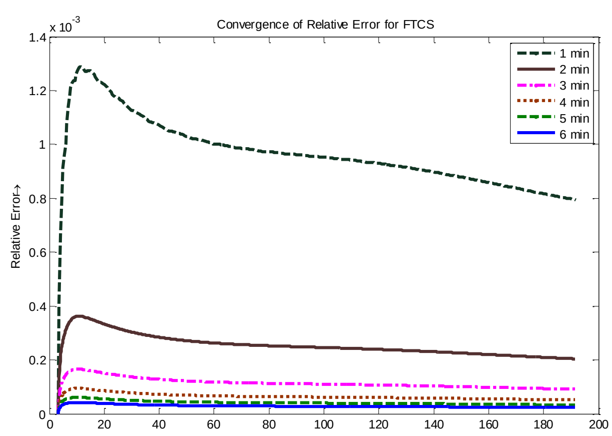

Figure 5.2 Rate of Numerical feature of Convergence

The following figure 5.3 shows the comparison of relative errors for the both schemes.

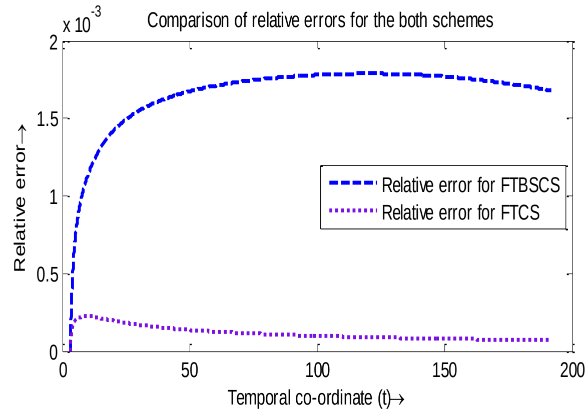

Figure 5.3 Comparison of relative errors for the both schemes

6. Conclusion

We have presented stability analysis, analytical solutions and numerical solutions for 1D advection diffusion equation with an initial condition and Neumann boundary conditions. Numerical experiment is presented graphically. The analytical result is used for code validation and for error comparison of both schemes. In addition, it is used to study the effect of step size on the accuracy of solutions. The results shown in Figure 5.1 – 5.3 are the error terms as defined above at time level [1, 6]. Two points to emphasize with regard to Figure 5.1-5.3 are: (1) for this application, the FTCS scheme has minimum error in comparison with FTBSCS scheme, and the amount of error is decreased for the both schemes as the solution is marched in time. This error reduction is due to a decrease in the influence of the initial data.

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